2-DIMENSIONAL TOPOLOGICAL QUANTUM FIELD THEORIES & COMMUTATIVE FROBENIUS ALGEBRAS

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Abstract. In this brief survey we study the categorical equivalence between 2-dimensional topological quantum field theories and commutative Frobenius algebras. To that end, we develop some of the main tools to understand the main result, which has an extensive use of the theory of manifolds and categories.

1. Monoidal Categories

The theory of monoidal categories plays a main role in our study of topological quantum field theories and shall be used extensively. For a more thorough analysis see [Eti+16]. If the reader is not acquainted with a working knowledge of category theory, a good resource is [Bor08].

Definition 1.1. A *monoidal category* is a tuple $(\mathtt{M}, \otimes, 1, \alpha, \lambda, \rho)$ consisting of:

- A *category* $\mathtt{M}$.
- A *bifunctor* $\otimes \colon \mathtt{M} \times \mathtt{M} \longrightarrow \mathtt{M}$
- A distinguished object $1 \in \mathtt{M}$ that is *unitary* with respect to $\otimes$, that is:

$$m \otimes 1 = m = 1 \otimes m$$

  for any object $m \in \mathtt{M}$.
- A *natural isomorphism*

$$\alpha \colon (- \otimes (- \otimes -)) \overset{\simeq}{\Longrightarrow} ((- \otimes -) \otimes -)$$

  called *associator*. We call α a natural isomorphism in the sense that given any triple of objects (a, b, c) of $\mathtt{M}$, the image

$$a \otimes (b \otimes c) \xrightarrow[\alpha(a,b,c)]{\simeq} (a \otimes b) \otimes c$$

  is an isomorphism in $\mathtt{M}$.
- Two *natural isomorphisms*

$$\lambda \colon (1 \otimes -) \overset{\simeq}{\Longrightarrow} (-) \quad \text{and} \quad \rho \colon (- \otimes 1) \overset{\simeq}{\Longrightarrow} (-)$$

  called *left and right unitors*, respectively. In other words, given any object $a \in \mathtt{M}$ the arrows $\lambda a \colon 1 \otimes a \xrightarrow{\simeq} a$ and $\rho a \colon a \otimes 1 \xrightarrow{\simeq} a$ are isomorphisms in $\mathtt{M}$.

This data should satisfy the following two conditions:

- (Triangle identity) Given any pair (a, b) of objects in $\mathtt{M}$, the diagram

$$\begin{array}{ccc} a \otimes (1 \otimes b) & \xrightarrow{\alpha(a,1,b)} & (a \otimes 1) \otimes b \\ & {\scriptstyle \mathrm{id}_a \otimes \rho b} \searrow \quad \swarrow {\scriptstyle \lambda a \otimes \mathrm{id}_b} & \\ & a \otimes b & \end{array}$$

  commutes in $\mathtt{M}$.

- (Pentagon identity) Given any tuple (a, b, c, d) of objects in $\mathtt{M}$, the diagram

$$
\begin{array}{ccccc}
 & & (a \otimes b) \otimes (c \otimes d) & & \\
 & \nearrow^{\alpha(a,b,c\otimes d)} & & \searrow^{\alpha(a\otimes b,c,d)} & \\
a \otimes (b \otimes (c \otimes d)) & & & & ((a \otimes b) \otimes c) \otimes d \\
\Big\downarrow{\scriptstyle \mathrm{id}_a \otimes \alpha(b,c,d)} & & & & \Big\uparrow{\scriptstyle \alpha(a,b,c)\otimes \mathrm{id}_d} \\
a \otimes ((b \otimes c) \otimes d) & & \xrightarrow[\alpha(a,b\otimes c,d)]{} & & (a \otimes (b \otimes c)) \otimes d
\end{array}
$$

is commutative in $\mathtt{M}$.

The tuple $(\mathtt{M}, \otimes, 1, \alpha, \lambda, \rho)$ is said to be a *strict monoidal category* if the three natural isomorphisms α, λ and ρ are naturally isomorphic to the identity. If this is the case, we shall refer to the category simply by the triple $(\mathtt{M}, \otimes, 1)$.

This monoidal structure can be also be carried to functors and natural transformations:

Definition 1.2 (Monoidal functor)**.** Let $(\mathtt{M}, \otimes, 1, \alpha, \lambda, \rho)$ and $(\mathtt{N}, \widehat{\otimes}, \widehat{1}, \widehat{\alpha}, \widehat{\lambda}, \widehat{\rho})$ be two (strict) monoidal categories. We say that a functor $F\colon \mathtt{M} \to \mathtt{N}$ is a (*strict*) *monoidal functor* if it preserves the actions of the natural isomorphisms. To put concretely, we have:

- The unit of $\mathtt{M}$ is mapped to the unit of $\mathtt{N}$, that is, $F1 = \widehat{1}$.
- For any $a \in \mathtt{M}$ one has $F(\lambda a) = \widehat{\lambda}(Fa)$ and $F(\rho a) = \widehat{\rho}(Fa)$.
- For any pair (a, b) of objects in $\mathtt{M}$ there exists an isomorphism $F(a \otimes b) \simeq Fa \widehat{\otimes} Fb$ in $\mathtt{N}$. In the strict case the isomorphism is replaced by an equality.
- For any triple (a, b, c) of objects in $\mathtt{M}$ we have $F\alpha(a, b, c) = \widehat{\alpha}(Fa, Fb, Fc)$.
- For every two maps f and g in $\mathtt{M}$ there exists an isomorphism $F(f \otimes g) \simeq Ff \widehat{\otimes} Fg$ in $\mathtt{N}$. As before, in the strict case the isomorphism is replaced by an equality.

Definition 1.3 (Monoidal natural transformation)**.** Let $(\mathtt{M}, \otimes, 1, \alpha, \lambda, \rho)$ and $(\mathtt{N}, \widehat{\otimes}, \widehat{1}, \widehat{\alpha}, \widehat{\lambda}, \widehat{\rho})$ be two (strict) monoidal categories, and consider a pair of parallel (strict) monoidal functors $F, G\colon \mathtt{M} \rightrightarrows \mathtt{N}$. A natural transformation $\eta\colon F \Rightarrow G$ is said to be *monoidal* if $\eta_1 = \widehat{1}$, and for any pair of objects $a, b \in \mathtt{M}$ the diagram

$$
\begin{array}{ccc}
F(a \otimes b) & \xrightarrow{\eta_{a\otimes b}} & G(a \otimes b) \\
{\scriptstyle \simeq}\Big\downarrow & & \Big\downarrow{\scriptstyle \simeq} \\
Fa \widehat{\otimes} Fb & \xrightarrow[\eta_a \widehat{\otimes} \eta_b]{} & Ga \widehat{\otimes} Gb
\end{array}
$$

commutes in the monoidal category $\mathtt{N}$.

The following theorem allows one to always work with a strictified version of a given monoidal category. Its proof, however, is extensive and would not fit in this short essay. For a proof, the curious reader can refer to [Gei22].

Theorem 1.4. Every monoidal category is *monoidally equivalent* to a *strict* monoidal category.

As in algebraic contexts, we can also find monoids inside of a given monoidal category.

Definition 1.5. Let $(\mathtt{M}, \otimes, 1, \alpha, \lambda, \rho)$ be a monoidal category. We define the following objects:

(a) A *monoid* in $\mathtt{M}$ is a triple (m, μ, θ) where we have an object $m \in \mathtt{M}$, a bifunctor $\mu\colon m \otimes m \to m$, referred to as a *multiplication*, and a functor $\theta\colon 1 \to m$, called *unit*, such that both diagrams

$$\begin{array}{ccccc} m\otimes(m\otimes m) & \xrightarrow{\alpha(m,m,m)} & (m\otimes m)\otimes m & \xrightarrow{\mu\otimes\mathrm{id}_m} & m\otimes m \\ \downarrow{\scriptstyle \mathrm{id}_m\otimes\mu} & & & & \downarrow{\scriptstyle\mu} \\ m\otimes m & & \xrightarrow{\qquad\mu\qquad} & & m \end{array} \qquad \begin{array}{ccccc} 1\otimes m & \xrightarrow{\theta\otimes\mathrm{id}_m} & m\otimes m & \xleftarrow{\mathrm{id}_m\otimes\theta} & m\otimes 1 \\ & {\scriptstyle\rho}\searrow & \downarrow{\scriptstyle\mu} & \swarrow{\scriptstyle\lambda} & \\ & & m & & \end{array}$$

commute in $\mathtt{M}$. A morphism of monoids $\phi\colon (m, \mu, \theta) \to (m', \mu', \theta')$ is a morphism $\phi\colon m \to m'$ in $\mathtt{M}$ satisfying $\phi\mu = \mu'(\phi \otimes \phi)$, and $\phi\theta = \theta'$. We then define the subcategory $\mathtt{Mon}_{\mathtt{M}}$ of $\mathtt{M}$ composed of monoidal objects in $\mathtt{M}$.

(b) A *comonoid* in $\mathtt{M}$ is a triple (c, κ, σ) where c is an object of $\mathtt{M}$, we have a bifunctor $\kappa\colon c \to c \otimes c$, called *comultiplication*, and a functor $\sigma\colon c \to 1$, called *counit*, such that both diagrams

$$\begin{array}{ccccc} c\otimes(c\otimes c) & \xleftarrow{\alpha(c,c,c)^{-1}} & (c\otimes c)\otimes c & \xleftarrow{\kappa\otimes\mathrm{id}_c} & c\otimes c \\ \uparrow{\scriptstyle \mathrm{id}_c\otimes\kappa} & & & & \uparrow{\scriptstyle\kappa} \\ c\otimes c & & \xleftarrow{\qquad\kappa\qquad} & & c \end{array} \qquad \begin{array}{ccccc} 1\otimes c & \xleftarrow{\sigma\otimes\mathrm{id}_c} & c\otimes c & \xrightarrow{\mathrm{id}_c\otimes\sigma} & c\otimes 1 \\ & {\scriptstyle\rho^{-1}}\nwarrow & \uparrow{\scriptstyle\kappa} & \nearrow{\scriptstyle\lambda^{-1}} & \\ & & c & & \end{array}$$

commute in $\mathtt{M}$. A morphism of comonoids $\psi\colon (c, \kappa, \sigma) \to (c', \kappa', \sigma')$ is a morphism $\psi\colon c \to c'$ in $\mathtt{M}$ satisfying $\kappa'\psi = (\psi \otimes \psi)\kappa$, and $\sigma = \sigma'\psi$. We then define the subcategory $\mathtt{coMon}_{\mathtt{M}}$ of $\mathtt{M}$ composed of comonoidal objects in $\mathtt{M}$.

An important example of the later is that of algebras and coalgebras in the category of vector spaces—those are monoids and comonoids, respectively. These will appear later in the text.

Monoids, groups and the like cannot live without the core concept of actions, so now we also define them in this abstract context.

Definition 1.6 (Monoid actions)**.** Let $(\mathtt{M}, \otimes, 1)$ be a monoidal category, and $(m, \mu, \theta) \in \mathtt{Mon}_{\mathtt{M}}$. A *left-action* of the monoid (m, μ, θ) on an object $a \in \mathtt{M}$ is a bifunctor $\sigma\colon m \otimes a \to a$ such that

$$\begin{array}{ccccccc} m\otimes(m\otimes a) & \xrightarrow{\alpha(m,m,a)} & (m\otimes m)\otimes a & \xrightarrow{\mu\otimes\mathrm{id}_a} & m\otimes a & \xleftarrow{\theta\otimes\mathrm{id}_a} & 1\otimes a \\ \downarrow{\scriptstyle \mathrm{id}_m\otimes\sigma} & & & & \downarrow{\scriptstyle\sigma} & \swarrow{\scriptstyle\lambda} & \\ m\otimes a & & \xrightarrow{\qquad\sigma\qquad} & & a & & \end{array}$$

commutes in $\mathtt{M}$. Right-actions are defined analogously.

Given any two left-actions $\sigma\colon m \otimes a \to a$ and $\lambda\colon m \otimes b \to b$, we define a *morphism of left-actions* $\phi\colon \sigma \to \lambda$ to be an arrow $\phi\colon a \to b$ in $\mathtt{M}$ such that the square

$$\begin{array}{ccc} m\otimes a & \xrightarrow{\mathrm{id}_m\otimes\phi} & m\otimes b \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\lambda} \\ a & \xrightarrow{\quad\phi\quad} & b \end{array}$$

commutes in $\mathtt{M}$. With these notions we are able to define two categories $\mathtt{rActMon}_{(\mathtt{M},m)}$ and $\mathtt{lActMon}_{(\mathtt{M},m)}$, composed of right and left actions of m on objects of $\mathtt{M}$, respectively, and morphisms between them.

2. Braided & Symmetric Monoidal Categories

So far, we've only talked about monoidal structures that have a non-commutative associated product. In our context we would also like to understand the situations where commutativity is allowed. For instance, in the category of vector spaces we have a natural isomorphism

$V \otimes W \simeq W \otimes V$ for any pair $V, W \in \mathtt{Vect}_k$. To that end, we define the concept of braiding, and associated to it the notion of braided monoidal categories.

Definition 2.1 (Braiding)**.** Given a monoidal category $(\mathtt{M}, \otimes, 1, \alpha, \lambda, \rho)$, we define a *braiding* of $\mathtt{M}$ to be a natural isomorphism

$$\gamma\colon (- \otimes -') \overset{\cong}{\Longrightarrow} (-' \otimes -),$$

that is coherent with associativity and unitors of $\mathtt{M}$, in the sense that the diagrams

$$\begin{array}{ccc}
(a \otimes b) \otimes c & \xrightarrow{\gamma_{(a\otimes b,c)}} & c \otimes (a \otimes b) \\
\Big\downarrow {\scriptstyle \alpha^{-1}(a,b,c)} & & \Big\downarrow {\scriptstyle \alpha(c,a,b)} \\
a \otimes (b \otimes c) & & (c \otimes a) \otimes b \\
\Big\downarrow {\scriptstyle \mathrm{id}_a \otimes \gamma_{(b,c)}} & & \Big\downarrow {\scriptstyle \gamma_{(c,a)} \otimes \mathrm{id}_b} \\
a \otimes (c \otimes b) & \xrightarrow[\alpha(a,c,b)]{} & (a \otimes c) \otimes b
\end{array}
\qquad
\begin{array}{ccc}
a \otimes (b \otimes c) & \xrightarrow{\gamma_{(a,b\otimes c)}} & (b \otimes c) \otimes a \\
\Big\downarrow {\scriptstyle \alpha(a,b,c)} & & \Big\downarrow {\scriptstyle \alpha(b,c,a)^{-1}} \\
(a \otimes b) \otimes c & & b \otimes (c \otimes a) \\
\Big\downarrow {\scriptstyle \gamma_{(a,b)} \otimes \mathrm{id}_c} & & \Big\downarrow {\scriptstyle \gamma_{(c,a)} \otimes \mathrm{id}_b} \\
(b \otimes a) \otimes c & \xrightarrow[\alpha(b,a,c)]{} & b \otimes (a \otimes c)
\end{array}$$

$$\begin{array}{ccc}
a \otimes 1 & \xrightarrow{\gamma_{(a,1)}} & 1 \otimes a \\
& {\scriptstyle \lambda} \searrow \quad \swarrow {\scriptstyle \rho} & \\
& a &
\end{array}$$

should commute for all triples (a, b, c) of objects of $\mathtt{M}$. Naturally, we say that a monoidal category is braided if it is associated with a braiding.

A really important concept for us will be that of functors between braided monoidal categories, they will play a central role in the last sections of this essay.

Definition 2.2 (Braided monoidal functor)**.** A monoidal functor $F\colon (\mathtt{A}, \gamma) \to (\mathtt{B}, \widehat{\gamma})$ between braided monoidal categories is said to be a *braided monoidal functor* if for every pair of objects $a, b \in \mathtt{A}$ the braiding coherence square

$$\begin{array}{ccc}
Fa \otimes Fb & \xrightarrow{\widehat{\gamma}} & Fb \otimes Fa \\
{\scriptstyle \simeq} \Big\downarrow & & \Big\downarrow {\scriptstyle \simeq} \\
F(a \otimes b) & \xrightarrow[F\gamma]{} & F(b \otimes a)
\end{array}$$

commutes in $\mathtt{B}$.

Now that we have both objects and functors between them, we may define a category $\mathtt{BrMonCat}$ composed of braided monoidal categories and braided monoidal functors between them. We can further restrict the objects of this category to obtain an even better behaved category:

Definition 2.3 (Symmetric monoidal category)**.** A braided monoidal category $(\mathtt{M}, \gamma)$ is said to be *symmetric* if for any two $a, b \in \mathtt{M}$ the triangle

$$\begin{array}{ccc}
a \otimes b & \xrightarrow{\mathrm{id}_{a\otimes b}} & a \otimes b \\
& {\scriptstyle \gamma_{(b,a)}} \nwarrow \quad \swarrow {\scriptstyle \gamma_{(a,b)}} & \\
& b \otimes a &
\end{array}$$

commutes in $\mathtt{M}$. A morphism between symmetric monoidal categories is a braided monoidal functor between them.

Example 2.4. One of the most important symmetric monoidal categories in our context will be $(\mathtt{Vect}_k, k, \otimes)$, the category of k-vector spaces, where k is the unitary object and the tensor product plays the role of product in the category.

3. Cobordisms

We now move a bit from the categorical madness and delve into yet another idea central to topological quantum field theories: cobordisms between manifolds.

3.1. **Unoriented Cobordisms.**

Definition 3.1 (Unoriented cobordism)**.** Given a pair Σ_0 and Σ_1 of smooth compact $(n-1)$-manifolds without boundary, we define a *cobordism between* Σ_0 *and* Σ_1 to be a smooth compact n-manifold M whose boundary is $\partial M = \Sigma_0 \amalg \Sigma_1$. We thus call the manifolds Σ_0 and Σ_1 *cobordant*.

Example 3.2. Two interesting cobordisms are formed from the circle to the empty manifold, which shall be poetically called *death of a circle*, and from the empty manifold to the circle, so called *birth of a circle*. These are pictured as follows:

As an example of how cobordisms play in the wild, we have the following lemma—which investigates the situation for 0-manifolds.

Lemma 3.3 (Cobordant zero and one dimensional manifolds)**.** Two given compact 0-manifolds without boundary are cobordant if and only if they have the same number of points modulo 2. Moreover, any two compact 1-manifolds without boundary are cobordant.

Proof. Let's consider the case of a pair of 0-manifolds Σ_0 and Σ_1. Notice that since every pair of points can be connected by a smooth curve, and every 1-manifold with boundary has an even number of boundary points[1], it follows that Σ_0 and Σ_1 are cobordant if and only if the disjoint union $\Sigma_0 \amalg \Sigma_1$ has an even number of points.

For the second statement, one should recall that a compact 1-manifold is the disjoint union of circles. Then we can choose one of the manifolds to attach copies of the death of a circle cobordism for each of its circles, and attach birth of a circle cobordisms for each of its respective circles of the other 1-manifold. This construction yields a cobordism between them. ♮

3.2. **Oriented Cobordisms.**

Consider the following setup: let Σ be a closed submanifold of M with codimension 1, where $\dim M = n$. Assume both manifolds to be oriented. Our goal will be to define the concept of orientation in a cobordism.

Definition 3.4 (Positive normal)**.** Let $[v_1, \ldots, v_{n-1}]$ be a positive basis for $T_x\Sigma$ for any given point $x \in \Sigma$. We say that a tangent vector $v \in T_xM$ is a *positive normal* if the induced basis $[v_1, \ldots, v_{n-1}, v]$ for T_xM is positive.

Positive normals grant the possibility to distinguish boundaries of a manifold by choosing something analogous to a time arrow.

[1] This is due to the fact that 1-manifolds are C^∞-isomorphic to a finite disjoint union of circles or intervals (see the appendix of [Mil97]).

Definition 3.5 (In and out boundaries)**.** If Σ is a connected component of ∂M, we call Σ an *in-boundary* if a positive normal points inwards relative to M, and otherwise an *out-boundary*—when a positive normal points outward relative to M.

The notion of an in and out boundary allows us to define the notion of an oriented cobordism. From now on, *a cobordism will always mean an oriented* one, unless stated otherwise.

Definition 3.6 (Oriented cobordism)**.** Let Σ_{in} and Σ_{out} be compact $(n-1)$-manifolds without boundary. We define an *oriented cobordism* between them to be a triple $(M, \iota_{\text{in}}, \iota_{\text{out}})$, where M is a smooth compact oriented n-manifold, and arrows

$$\Sigma_{\text{in}} \xrightarrow{\iota_{\text{in}}} M \xleftarrow{\iota_{\text{out}}} \Sigma_{\text{out}}$$

which are C^∞-isomorphisms when restricted to the in and out boundary of M, respectively. We shall denote the oriented cobordism M as an arrow $M\colon \Sigma_{\text{in}} \Rightarrow \Sigma_{\text{out}}$.

To give a better explanation to what was meant by the idea of choosing a time arrow, notice that given a cobordism one can slice the resulting manifold with a parametrization $I := [0,1]$, where at 0 one has a slice containing the in-boundaries, while at 1 we have a slice forming the out-boundary. This idea can come in handy for physical analogies.

Example 3.7 (Cylinder cobordism)**.** An important example of cobordism is that of a cylinder—it will play the role of an identity morphism in the category of cobordisms that we have yet to define. Let Σ be a compact oriented $(n-1)$-manifold without boundary. We define the *cylinder cobordism* of Σ to be the triple $(\Sigma \times I, \iota_{\text{in}}, \iota_{\text{out}})$, where we have canonical inclusions

$$\Sigma \times 0 \underset{\simeq}{\xleftarrow{\iota_{\text{in}}}} \Sigma \underset{\simeq}{\xhookrightarrow{\iota_{\text{out}}}} \Sigma \times 1$$

that is, the cobordism from Σ to Σ itself.

In order to distinguish between two cobordisms, it is important to know when two are equivalent or not. We address that as follows:

Definition 3.8 (Equivalence of cobordisms)**.** Given two cobordisms $M, N\colon \Sigma_{\text{in}} \Rightarrow \Sigma_{\text{out}}$, we say that M is *equivalent* to the cobordism N if there exists an orientation-preserving C^∞-isomorphism $\phi\colon M \xrightarrow{\simeq} N$ such that the following diagram commutes in `Man`:

$$\begin{array}{ccccc} & \nearrow & N & \nwarrow & \\ \Sigma_{\text{in}} & & \phi \uparrow \simeq & & \Sigma_{\text{out}} \\ & \searrow & M & \swarrow & \end{array}$$

4. Elements of Morse Theory

An interesting tool for dealing with cobordisms is provided by Morse theory, which we'll only scrap the surface and merely give some pertinent definitions for our discussions.

Definition 4.1. Let $f\colon M \to I$ be a C^∞-morphism, and $p \in M$ be a critical point of f. We call p a *non-degenerate* point if there exists a chart about p for which the local Hessian of f is invertible. Furthermore, we define the *index of f at p* to be the number of *negative eigenvalues* of the local Hessian.

Definition 4.2 (Morse maps)**.** Let M be a smooth manifold. We say that a C^∞-morphism $f: M \to I$ is a *Morse map* if every critical point of f is non-degenerate. If it happens to be the case that M is a manifold with boundary, we shall further require that $f^{-1}(\partial I) = \partial M$ and that the boundary points $\partial I = \{0, 1\}$ are regular values of f—ensuring that ∂M has no critical points.

The existence of Morse maps is ensured by the following theorem, which can be found in [Hir76]:

Theorem 4.3. For any manifold M and integer $2 \leqslant r \leqslant \infty$, the collection of Morse maps $M \to I$ is dense in $C^r(M, I)$.

4.1. **Decomposing & Gluing Cobordisms.** We'll use the concept of gluing of spaces in order to provide the concept of composition of cobordisms. For that, we have the following definition:

Definition 4.4 (Gluing)**.** Let $f: X \to Y$ and $g: X \to Z$ be topological morphisms. We define the *gluing of* Y *and* Z *along* X to be the pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & \ulcorner & \downarrow \\ Z & \longrightarrow & Y \amalg_X Z \end{array}$$

Explicitly, $Y \amalg_X Z$ is the quotient space of $Y \amalg Z$ where $y \sim z$ if and only if there exists a common $x \in X$ such that $fx = y$ and $gx = z$. For notational purposes, the gluing of Y and Z along X can also be denoted by YZ when X is implicitly understood.

Given a cobordism $M: \Sigma_0 \Rightarrow \Sigma_1$, one can think how to slice M so that we get a smooth submanifold $\Sigma_t \subseteq M$ dividing M into two. The first part should contain all in-boundaries of M, while the second should contain each out-boundary of M. To that end, take a Morse map $f: M \to I$ such that $f^{-1}(0) = \Sigma_0$ and $f^{-1}(1) = \Sigma_1$. We shall then define Σ_t as the preimage $f^{-1}(t) \subseteq M$, yielding two cobordisms

$$M_{[0,t]} := f^{-1}([0,t]): \Sigma_0 \Longrightarrow \Sigma_t \quad \text{and} \quad M_{[t,1]} := f^{-1}([t,1]): \Sigma_t \Longrightarrow \Sigma_1.$$

The following theorem can be found in [Hir76] (page 153).

Theorem 4.5 (Regular interval)**.** Let $M: \Sigma_0 \Rightarrow \Sigma_1$ be a cobordism, and $f: M \to I$ a Morse map admitting no critical points and such that $f^{-1}(0) = \Sigma_0$ and $f^{-1}(1) = \Sigma_1$. If $\pi: \Sigma_0 \times I \twoheadrightarrow I$ denotes the canonical projection, then there exists a C^∞-isomorphism $\phi: \Sigma_0 \times I \to M$ such that the following diagram commutes in `Man`:

$$\begin{array}{ccc} \Sigma_0 \times I & \underset{\simeq}{\xrightarrow{\phi}} & M \\ & {\scriptstyle \pi}\searrow & \downarrow{\scriptstyle f} \\ & & I \end{array}$$

A relevant consequence of the regular interval theorem is the following:

Lemma 4.6. Let $M: \Sigma_0 \Rightarrow \Sigma_1$ be a cobordism, and $f: M \to I$ a Morse map with $f^{-1}(0) = \Sigma_0$ and $f^{-1}(1) = \Sigma_1$. Then there exists $\varepsilon > 0$ and a *decomposition*

$$M = M_{[0,\varepsilon]} M_{[\varepsilon,1]}$$

such that $M_{[0,\varepsilon]}$ is C^∞-isomorphic to $\Sigma_0 \times I$. Analogously, there also exists a decomposition relating to the out-boundary Σ_1.

Proof. M being a manifold with boundary implies that f has no critical points at the in and out boundaries of M. Moreover, since f is a C^∞-morphism it follows that there must exist some pair $\varepsilon_1, \varepsilon_2 > 0$ for which $[0, \varepsilon_1]$ and $[1 - \varepsilon_2, 1]$ are both sets of regular point of f. We can then conclude that both restrictions $f|_{M_{[0,\varepsilon_1]}}$ and $f|_{M_{[1-\varepsilon_2,1]}}$ are maps lacking critical points, and thus satisfy the conditions to be Morse maps. Applying Theorem 4.5, we obtain that $M_{[0,\varepsilon_1]} \simeq \Sigma_0 \times I$, and $M_{[1-\varepsilon_2,1]} \simeq \Sigma_1 \times I$, and both claimed decomposition of M. ♮

5. The Category n-cob

We would now like to construct a *category of cobordisms* of a given dimension n. For that, our objects shall be compact oriented $(n-1)$-manifolds, but what about the arrows? A solution for this dilemma is to use *classes* of oriented cobordisms since we already know how to distinguish one from the other. In order for that to work, we shall define the idea of *composition* of cobordisms and show that if $M = M_1 M_0$ is a decomposition of a cobordism M, then the composition of M_0 and M_1 need to be M—that is, compositions and decompositions play nicely with each other.

Our first step will be to define a gluing of any two cobordisms. For that, let $M_0\colon \Sigma_0 \Rightarrow \Sigma_1$ and $M_1\colon \Sigma_1 \Rightarrow \Sigma_2$ be two cobordisms, and consider Morse maps $f_0\colon M_0 \to [0,1]$ and $f_1\colon M_1 \to [1,2]$. We'll try to relate the topological manifold $M_0 \amalg_{\Sigma_1} M_1$—which also comes with a continuous map $M_0 \amalg_{\Sigma_1} M_1 \to [0,2]$ induced by f_0 and f_1—to a cobordism of the form $\Sigma_0 \Rightarrow \Sigma_2$. Choose $\varepsilon > 0$ such that the intervals $[1-\varepsilon, 1]$ and $[1, 1+\varepsilon]$ are regular sets of f_0 and f_1, respectively. By Theorem 4.5 we conclude that there are C^∞-isomorphisms $M_{[1-\varepsilon,1]} \simeq \Sigma_1 \times [1-\varepsilon, 1]$ and $M_{[1,1+\varepsilon]} \simeq \Sigma_1 \times [1, 1+\varepsilon]$, thus there exists a *topological* isomorphism

$$M_{[1-\varepsilon,1]} \amalg_{\Sigma_1} M_{[1,1+\varepsilon]} \simeq \Sigma_1 \times [1-\varepsilon, 1+\varepsilon],$$

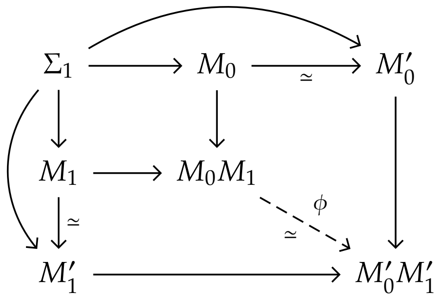
whose restriction to $M_{[1-\varepsilon,1]}$ and $M_{[1,1+\varepsilon]}$ is a C^∞-isomorphism. As a by-product one gets a smooth structure for the whole gluing space $M_0 \amalg M_1$, which is itself a *cobordism class*

$$M_0 \amalg M_1 \colon \Sigma_0 \Longrightarrow \Sigma_2.$$

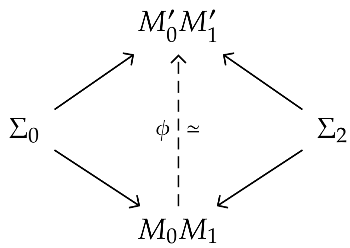
We stress that such cobordism indeed represents a *class* since the *choice* of the smooth structure *isn't canonical*.

Theorem 5.1. Let M_0 and M_1 be cobordisms such that Σ is the out-boundary of M_0 and the in-boundary of M_1. If α and β are two smooth structures for the gluing $M_0 M_1$ along Σ, there exists a C^∞-isomorphism $\phi\colon (M_0 M_1, \alpha) \xrightarrow{\simeq} (M_0 M_1, \beta)$ such that $\phi|_\Sigma = \mathrm{id}_\Sigma$.

An important direct consequence of this theorem is that the gluing of cobordisms has a unique smooth structure up to smooth isomorphism. Moreover, the smooth structure should not depend on the choice of representative of the cobordism class—and this is what we now show. To see this, consider any two cobordisms $M_0\colon \Sigma_0 \Rightarrow \Sigma_1$ and $M_1\colon \Sigma_1 \Rightarrow \Sigma_2$, and suppose there exists two cobordism equivalences $\phi_0\colon M_0 \xrightarrow{\simeq} M_0'$ and $\phi_1\colon M_1 \xrightarrow{\simeq} M_1'$—that is, we consider any other representatives M_0' and M_1'. Using the universal property of pushouts associated to the gluings $M_0 M_1$ and $M_0' M_1'$, we get the following commutative diagram in Top, where ϕ is a unique *topological* isomorphism:



The resulting map ϕ restricts to a C^∞-isomorphism between manifolds on each of its factors. This shows that ϕ is, in fact, an equivalence of cobordisms:



From now on we shall refer to cobordisms by their classes of equivalence.

With this we are finally ready to define precisely what we mean by the composition of cobordisms classes:

Definition 5.2 (Composition of cobordisms)**.** Consider cobordisms classes $M_0\colon \Sigma_0 \Rightarrow \Sigma_1$ and $M_1\colon \Sigma_1 \Rightarrow \Sigma_2$. We define the *composition cobordism* of M_0 with M_1 as the cobordism class $M_0M_1\colon \Sigma_0 \Rightarrow \Sigma_2$ given by the gluing of M_0 and M_1 along Σ_1.

Lemma 5.3. The composition of cobordisms is associative.

Proof. Consider classes of cobordisms $\Sigma_0 \overset{M_0}{\Longrightarrow} \Sigma_1 \overset{M_1}{\Longrightarrow} \Sigma_2 \overset{M_2}{\Longrightarrow} \Sigma_3$. To see that

$$(M_0M_1)M_2 = M_0(M_1M_2)$$

it suffices to observe that: in $(M_0M_1)M_2$ we first identify the common boundary Σ_1 of M_0 and M_1, and construct the smooth structure around Σ_1 as discussed before—further we identify Σ_2 in the same manner and obtain a gluing of $M_0 \amalg_{\Sigma_1} M_1$ with M_2 along Σ_2. Notice that in this process we merely affected the smooth structure around a small neighbourhood of Σ_1 and Σ_2, and since those two boundaries are disjoint, this process can be done so that the order in which we glue does not affect the resulting structure. ♮

Back to what we anticipated at Example 3.7, we can now say in what way the cylinder cobordisms act as an identity in the classes of cobordisms:

Lemma 5.4. Given any cobordism $M\colon \Sigma_0 \Rightarrow \Sigma_1$, if C_0 and C_1 denote the cylinder cobordisms of Σ_0 and Σ_1 respectively, then

$$C_0M = M = MC_1.$$

Now that we have objects and arrows that allow for composition, associativity, and identities, we can finally define our desired category. For each $n \in \mathbf{Z}_{>0}$ we'll denote by n-`cob` the category of n-dimensional cobordisms, whose objects are compact oriented $(n-1)$-manifolds without boundary, and morphisms are classes of cobordisms between them.

6. Monoidal Structure of n-cob

Notice that the coproduct given by the disjoint union in the category of manifolds can be extended to a bifunctor $\amalg\colon n\text{-}\mathtt{cob} \times n\text{-}\mathtt{cob} \longrightarrow n\text{-}\mathtt{cob}$ by associating each pair of manifolds Σ_0 and Σ_1 of n-cob with the oriented $(n-1)$-manifold $\Sigma_0 \amalg \Sigma_1$ whose orientation is the canonical one—so that inclusions are orientation preserving. Also, given two cobordisms $M\colon \Sigma_0 \Rightarrow \Sigma_1$ and $N\colon \Theta_0 \Rightarrow \Theta_1$ we naturally obtain a cobordism

$$M \amalg N\colon \Sigma_0 \amalg \Theta \Longrightarrow \Sigma_1 \amalg \Theta_1.$$

Denoting by $\varnothing_{n-1} \in n\text{-}\mathtt{cob}$ the empty $(n-1)$-manifold, we know that

$$\varnothing_{n-1} \amalg \Sigma \simeq \Sigma \simeq \Sigma \amalg \varnothing_{n-1}$$

for any $\Sigma \in n\text{-}\mathtt{cob}$. Furthermore, if $\varnothing_n\colon \varnothing_{n-1} \Rightarrow \varnothing_{n-1}$ denotes the empty n-cobordism, we also find that $M \amalg \varnothing_n = M = \varnothing_n \amalg M$ are the same cobordism classes. This shows that $\varnothing_{n-1}$ serves as a unit with respect to the bifunctor $\amalg$ in the category n-cob. Thus we've obtained a *monoidal structure* $(n\text{-}\mathtt{cob}, \amalg, \varnothing)$.

Back to the construction of cylinders, one can extend the ideas of Example 3.7 to isomorphism between manifolds. Consider the subcategory $\mathtt{C}$ of $\mathtt{Man}_{n-1}$, whose objects are compact oriented $(n-1)$-manifolds without boundary, and C^∞-isomorphisms between them. Let $f\colon \Sigma_0 \xrightarrow{\simeq} \Sigma_1$ be a morphism in $\mathtt{C}$, and define a cobordism $(\Sigma_0 \times I, i_0, f_1^{-1})$ where $i_0\colon \Sigma_0 \longrightarrow \Sigma_0 \times I$ maps Σ_0 to the in-boundary of $\Sigma_0 \times I$ while $f_1^{-1}\colon \Sigma_1 \longrightarrow \Sigma_0 \times I$ maps Σ_1 to the out-boundary of $\Sigma_0 \times I$ via $x \mapsto (f^{-1}x, 1)$. Analogously, we define a cobordism $(\Sigma_1 \times I, f, i_1)$. This construction is clear to make the diagram

$$\begin{array}{ccccc} & & \Sigma_1 \times I & & \\ & \overset{f}{\nearrow} & \uparrow & \overset{i_1}{\nwarrow} & \\ \Sigma_0 & & {\scriptstyle f\times \mathrm{id}_I}\ \big|\ {\scriptstyle\simeq} & & \Sigma_1 \\ & \underset{i_0}{\searrow} & & \underset{f^{-1}}{\swarrow} & \\ & & \Sigma_0 \times I & & \end{array}$$

commute, showing that $f \times \mathrm{id}_I$ is an isomorphism of cobordisms and hence $\Sigma_1 \times I \simeq \Sigma_0 \times I$—this cobordism class will be denoted by $\mathrm{Cyl}\, f$. This construction is functorial, which means that we can define a functor $\mathrm{Cyl}\colon \mathtt{C} \longrightarrow n\text{-}\mathtt{cob}$ where we map the objects via inclusions and $f \mapsto \mathrm{Cyl}\, f$ as described above. Indeed, given another arrow $g\colon \Sigma_1 \xrightarrow{\simeq} \Sigma_2$ of $\mathtt{C}$, one has $\mathrm{Cyl}(g)\,\mathrm{Cyl}(f) = \mathrm{Cyl}(gf)$.

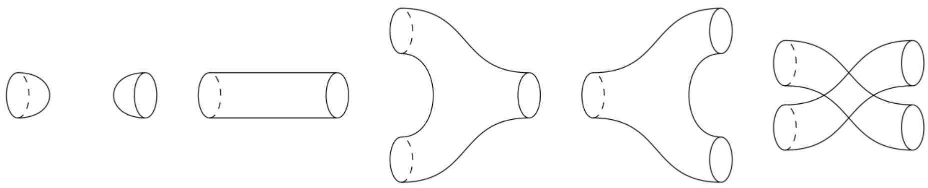
Since $\amalg$ is a coproduct in $\mathtt{Man}$, there exists a natural isomorphism $\gamma\colon (- \amalg -') \overset{\simeq}{\Longrightarrow} (-' \amalg -)$ in $\mathtt{Man}_{n-1}$. This natural transformation induces, via the cylinder construction, another natural isomorphism

$$T := \mathrm{Cyl} \circ \gamma\colon (- \amalg -') \overset{\simeq}{\Longrightarrow} (-' \amalg -),$$

this time in n-cob—which we shall refer to as the *twist cobordism*. The birth of T gives to n-cob the structure of a *symmetric monoidal category* $(n\text{-}\mathtt{cob}, \amalg, \varnothing, T)$.

6.1. **2-cob.** We shall be mostly interested in the case of two dimensional cobordisms, since for $n \geqslant 3$ the category n-cob has a highly non-trivial classification. We won't prove the classification theorem for 2-cob, which can be found in both [Koc03; Gei22] but nonetheless we at least state it below. An important part of the proof is that any pair of 2-dimensional cobordisms are members of the same isomorphism class if and only if they have the same number of connected components, that is, disjoint circles. This allows us to see the skeleton of 2-cob as $\mathbf{N}$, where each number $m \in \mathbf{N}$ denotes an isomorphism class whose members have a total of m connected components.

Theorem 6.1 (Classification of 2-cob)**.** Every 2-dimensional cobordism can be decomposed into the product of the following set of generators:



Those generators are named, from left to right, the *cup*, *cap*, *cylinder*, *copants*, *pants* and *twist* cobordisms.

In the skeletal view of 2-cob, the cup is a map $1 \Rightarrow 0$, the cap is the map $0 \Rightarrow 1$, the cylinder is $1 \Rightarrow 1$, copants give $2 \Rightarrow 1$, pants correspond to $1 \Rightarrow 2$, and the twist is a map $2 \overset{\cong}{\Rightarrow} 2$. An extensive number of relations between these generators can be found in theorem 2.40 of [Gei22], which we ask the curious reader to take a look.

7. Frobenius Algebras

We now introduce the second main character of our study: Frobenius algebras.

Definition 7.1 (Frobenius algebra)**.** Let $(\mathtt{M}, \otimes, 1)$ be a monoidal category, and $A \in \mathtt{M}$ be an object. A tuple $(A, \mu, \eta, \delta, \varepsilon)$ is said to be a *Frobenius algebra* if it satisfies the following requirements:

- The triple (A, μ, η) is a monoid, while (A, δ, ε) is a comonoid—where μ, η, δ and ε are arrows of $\mathtt{M}$.
- The diagram, encoding the so called *Frobenius relation*, commutes in $\mathtt{M}$:

$$\begin{array}{ccccc} (A\otimes A)\otimes A & \xleftarrow{\delta\otimes \mathrm{id}_A} & A\otimes A & \xrightarrow{\mathrm{id}_A\otimes\delta} & A\otimes(A\otimes A) \\ \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\simeq} \\ & & A & & \\ & & \downarrow{\scriptstyle\delta} & & \\ A\otimes(A\otimes A) & \xrightarrow[\mathrm{id}_A\otimes\mu]{} & A\otimes A & \xleftarrow[\mu\otimes\mathrm{id}_A]{} & (A\otimes A)\otimes A \end{array}$$

Pairings and copairings are structures that appear naturally in the context of vector spaces. Here we'll generalise this concept to Frobenius algebras:

Lemma 7.2. Let $(A, \mu, \eta, \delta, \varepsilon)$ be a Frobenius algebra in $(\mathtt{M}, \otimes, 1)$. There exists morphisms $\beta\colon A \otimes A \to 1$ and $\theta\colon 1 \to A \otimes A$—respectively called *pairing* and *copairing*—such that the following two diagrams commute in $\mathtt{M}$:

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{\mu\otimes\mathrm{id}_A} & A\otimes A \\ {\scriptstyle\mathrm{id}_A\otimes\mu}\downarrow & & \downarrow{\scriptstyle\beta} \\ A\otimes A & \xrightarrow[\beta]{} & 1 \end{array} \qquad \begin{array}{ccc} A\otimes A\otimes A & \xleftarrow{\mathrm{id}_A\otimes\delta} & A\otimes A \\ {\scriptstyle\delta\otimes\mathrm{id}_A}\uparrow & & \uparrow{\scriptstyle\theta} \\ A\otimes A & \xleftarrow[\theta]{} & 1 \end{array}$$

Proof. The pairing and copairing can be defined as $\beta := \varepsilon\mu$ and $\theta := \delta\eta$. Indeed, these definitions allow for the commutativity of the diagrams, since by Definition 1.5 we have

$$\beta(\mathrm{id}_A \otimes \mu) = (\varepsilon\mu)(\mathrm{id}_A \otimes \mu) = \varepsilon(\mu(\mathrm{id}_A \otimes \mu)) = \varepsilon(\mu(\mu \otimes \mathrm{id}_A)) = \beta(\mu \otimes \mathrm{id}_A),$$
$$(\delta \otimes \mathrm{id}_A)\theta = (\delta \otimes \mathrm{id}_A)(\delta\eta) = ((\delta \otimes \mathrm{id}_A)\delta)\eta = ((\mathrm{id}_A \otimes \delta)\delta)\eta = (\mathrm{id}_A \otimes \delta)\theta.$$

$\natural$

Our choice of pairing β and copairing θ in the last proof is, however, not unique. This motivates for the further restriction of a pairing to that of a non-degenerate one: we say that $\beta\colon A \otimes A \to 1$ is a *non-degenerate pairing* if and only if there exists $\theta\colon 1 \to A \otimes A$—called a *non-degenerate copairing*—such that the diagram

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\beta \otimes \mathrm{id}_A} & 1 \otimes A \simeq A \\ {\scriptstyle \mathrm{id}_A \otimes \theta}\uparrow & \nearrow_{\mathrm{id}_A} & \\ A \simeq A \otimes 1 & & \end{array} \tag{1}$$

commutes in the ambient monoidal category $\mathtt{M}$. As shown in [Gei22] lemma 3.21, given a non-degenerate pairing β, its corresponding non-degenerate copairing θ is unique up to isomorphism. In fact, the choice of $\beta = \varepsilon\mu$ and $\theta = \delta\eta$ satisfies our non-degeneracy condition.

In our context we shall be mostly concerned with the case where the underlying monoidal category of a Frobenius algebra is either *braided* or *symmetric*. A crucial structure for our main theorem is that of commutative Frobenius algebras, which we now define.

Definition 7.3 (Commutative Frobenius algebra)**.** We say that a Frobenius algebra $(A, \mu, \eta, \delta, \varepsilon)$ with an underlying braided monoidal category $(\mathtt{M}, \otimes, 1, \gamma)$ is *commutative* if the following diagram commutes in $\mathtt{M}$:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\gamma} & A \otimes A \\ & {\scriptstyle \mu}\searrow & \downarrow{\scriptstyle \mu} \\ & & A \end{array}$$

Equivalently one could define A to be commutative if and only if $\delta\gamma = \delta$ (see lemma 3.27 of [Gei22]).

In order to define a category consisting of Frobenius algebras, we need first consider how to construct morphisms between such objects. Given any two Frobenius algebras A and A' in a symmetric monoidal category $(\mathtt{M}, \otimes, 1, \gamma)$, we shall define a *Frobenius algebra morphism* $f\colon A \to A'$ to be a map that is both a monoid and comonoid morphism.

With this definition at hand, we can define the category $\mathtt{Frob}_{\mathtt{M}}$ of Frobenius algebras of the symmetric monoidal category $\mathtt{M}$. An important full subcategory is that of commutative Frobenius algebras, which we'll denote by $\mathtt{cFrob}_{\mathtt{M}}$.

The particular case we'll be mostly interested in is that of Frobenius algebras in the symmetric monoidal category $\mathtt{Vect}_k$. As already noted early in the text, in this ambient monoids are k-algebras, while comonoids are k-coalgebras. Therefore, given a Frobenius algebra $(A, \mu, \eta, \delta, \varepsilon)$ over $\mathtt{Vect}_k$ there exists a non-degenerate pairing $\beta := \varepsilon\mu\colon A \otimes A \to k$ and non-degenerate copairing $\theta := \delta\eta\colon k \to A \otimes A$. From lemma 2.1.13 of [Koc03] we conclude that A must be a finite dimensional k-vector space. With this in hands we find that a Frobenius algebra over $\mathtt{Vect}_k$ is simply a finite dimensional k-algebra A equipped with an associative[2] non-degenerate pairing $A \otimes A \to k$.

8. Topological Quantum Field Theories

The end our our brief journey is nigh, and we'll now get to know a fraction of topological quantum field theory.

[2]Given a k-algebra R and a pairing $f\colon V \otimes W \to k$ where V is a right R-module and W is a left R-module, we say that f is *associative* if for any pair $x, y \in V$ and $r \in R$ one has $f(x \otimes (r \cdot y)) = f((x \cdot r) \otimes y)$.

Definition 8.1. An n-dimensional *topological quantum field theory* over a field k is a symmetric monoidal functor $Z\colon n\text{-}\mathtt{cob} \to \mathtt{Vect}_k$. The collection of n-dimensional TQFT's over a field k, and monoidal natural transformations between them, forms a category

$$n\text{-}\mathtt{TQFT}_k = \mathtt{SymMon}(n\text{-}\mathtt{cob}, \mathtt{Vect}_k).$$

The next theorem comes simultaneously as a great and bad news. Topological quantum field theories can only realise finite dimensional spaces, which can be great for computational purposes, but in general cannot withstand many important physical applications that demand infinite dimensions.

Theorem 8.2. Let $Z \in n\text{-}\mathtt{TQFT}_k$ be any topological quantum field theory. Then the image of Z is a k-vector space equipped with a non-degenerate pairing—hence finite dimensional.

Proof. Let $\Sigma \in n\text{-}\mathtt{cob}$ be any object, and denote by $\overline{\Sigma} \in n\text{-}\mathtt{cob}$ the manifold with opposite orientation. Consider cobordisms $M\colon \Sigma \amalg \overline{\Sigma} \Rightarrow \varnothing_{n-1}$ and $N\colon \varnothing_{n-1} \Rightarrow \overline{\Sigma} \amalg \Sigma$. Since Z is a symmetric monoidal functor, these cobordisms induce k-linear morphisms $ZM\colon Z\Sigma \otimes Z\overline{\Sigma} \to k$ and $ZN\colon k \to Z\overline{\Sigma} \otimes Z\Sigma$. Notice that we can decompose the cylinder $\Sigma \times I$ as

$$\Sigma \times I \simeq \big((\Sigma \times I) \amalg N\big)\big(M \amalg (\Sigma \times I)\big),$$

thus by applying Z we obtain the equality

$$\mathrm{id}_{Z\Sigma} = \big(ZM \otimes \mathrm{id}_{Z\Sigma}\big)\big(\mathrm{id}_{Z\Sigma} \otimes ZN\big)$$

which is exactly the condition of commutativity imposed by Eq. (1)—thus ZM is a non-degenerate pairing and ZN its respective non-degenerate copairing. ♮

We know from Theorem 6.1 that objects of 2-cob are disjoint unions of circles and can be uniquely assigned to natural numbers $n \in \mathbf{N}$ corresponding to $\amalg^n S^1$. Since a TQFT Z is a monoidal functor, one has $Z(\amalg^n S^1) = (ZS^1)^{\otimes n}$. From this we can see that in order to understand how a 2-dimensional topological quantum field theory acts on a 2-dimensional cobordism it is sufficient to understand its action on a circle. To that end, we shall define more two structures.

Definition 8.3 (Free monoidal category over a Frobenius algebra)**.** Let $(\chi, \otimes, 0)$ be a monoidal category with a skeleton generated by 1—that is, the objects of χ are of the form $n = 1^{\otimes n}$— and whose morphisms are generated by arrows $\mu\colon 2 \to 1$, $\delta\colon 1 \to 2$, $\eta\colon 0 \to 1$, and $\varepsilon\colon 1 \to 0$. Moreover, we impose that the following relations are satisfied:

- Commutativity: $\mu(\mathrm{id} \otimes \eta) = \mathrm{id} = \mu(\eta \otimes \mathrm{id})$.
- Cocommutativity: $(\mathrm{id} \otimes \varepsilon)\delta = \mathrm{id} = (\varepsilon \otimes \mathrm{id})\delta$.
- Frobenius relations: $(\mathrm{id} \otimes \mu)(\delta \otimes \mathrm{id}) = \delta\mu = (\mu \otimes \mathrm{id})(\mathrm{id} \otimes \delta)$.

That is, the tuple $(1, \mu, \eta, \delta, \varepsilon)$ is a Frobenius algebra. With these conditions being satisfied, we shall call χ a *free monoidal category over the Frobenius algebra* 1.

As an immediate consequence of the imposed relations, we find that they also obey:

- Associativity: $\mu(\mathrm{id} \otimes \mu) = \mu(\mu \otimes \mathrm{id})$.
- Coassociativity: $(\delta \otimes \mathrm{id})\delta = (\mathrm{id} \otimes \delta)\delta$.

This shows a strong relationship to the properties of the generator S^1 and 2-cob. We still however lack any kind of twist morphism. This will be solved by the next definition.

Definition 8.4 (Free symmetric monoidal category over a Frobenius algebra)**.** Let $(\chi, \otimes, 0)$ be a free monoidal category over a Frobenius algebra 1. If we equip χ with a braiding $\gamma\colon 2 \xrightarrow{\simeq} 2$ and impose relations $\mu\gamma = \mu$ and $\gamma\delta = \delta$, it follows that $(1, \mu, \eta, \delta, \varepsilon)$ is a commutative Frobenius algebra, and we call χ a *free symmetric monoidal category over* 1.

With this definition we have that 2-cob is equivalent to any free symmetric monoidal category over a Frobenius algebra, where the braiding γ plays the role of the twist generator of 2-cob.

Since (symmetric) monoidal functors preserve the (symmetric) monoidal structure, in particular, for any $Z \in 2\text{-}\mathtt{TQFT}_k$, the image ZS^1 of the commutative Frobenius algebra $S^1 \in 2\text{-}\mathtt{cob}$ is itself a commutative Frobenius algebra over the symmetric monoidal category $\mathtt{Vect}_k$.

9. Equivalence Theorems

At last we reached the climax, we shall now show the connections between what we've been studying in the last few pages.

Theorem 9.1. Let χ be a free monoidal category over a Frobenius algebra 1. If $(\mathtt{M}, \otimes, e)$ is any monoidal category, then there exists a natural isomorphism

$$\mathtt{Mon}(\chi, \mathtt{M}) \simeq \mathtt{Frob}_{\mathtt{M}}.$$

Proof. Let $G\colon \chi \to \mathtt{M}$ be any monoidal functor, and define the Frobenius algebra $G(1) = F$. From the definition of G we know that any object $n \in \chi$ has an image $Gn = F^{\otimes n}$. For the arrow generators, we find their image under G satisfy all the needed conditions for the tuple $(F, G\mu, G\eta, G\delta, G\varepsilon)$ to be a Frobenius algebra.

On the other hand, if we start with the Frobenius algebra $F \in \mathtt{Frob}_{\mathtt{M}}$, we can associate to it the monoidal functor $\chi \to \mathtt{M}$ that maps 1 to F and the arrow generators corresponding to the important arrows of F as described in the last paragraph. With this we arrive exactly at the functor G.

Let $N, N' \in \mathtt{Mon}(\chi, \mathtt{M})$ be any two monoidal functors, and consider a monoidal natural transformation $\tau\colon N \Rightarrow N'$ between them. Denote by F and F' the Frobenius algebras $N(1)$ and $N'(1)$, respectively. For any object $n \in \chi$, the morphism τ_n is merely the n-th tensor product of the map $\tau_1\colon F \to F'$. From naturality of τ, the generators of χ induce the following commutative diagrams:

$$\begin{array}{ccc} F\otimes F & \xrightarrow{\tau_2} & F'\otimes F' \\ {\scriptstyle N\mu}\downarrow & & \downarrow{\scriptstyle N'\mu} \\ F & \xrightarrow[\tau_1]{} & F' \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{\tau_1} & F' \\ {\scriptstyle N\eta}\uparrow & & \uparrow{\scriptstyle N'\eta} \\ e & \overset{}{\underset{\tau_0}{=\!=}} & e \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{\tau_1} & F' \\ {\scriptstyle N\delta}\downarrow & & \downarrow{\scriptstyle N'\delta} \\ F\otimes F & \xrightarrow[\tau_2]{} & F'\otimes F' \end{array} \qquad \begin{array}{ccc} e & \overset{\tau_0}{=\!=} & e \\ {\scriptstyle N\varepsilon}\uparrow & & \uparrow{\scriptstyle N'\varepsilon} \\ F & \xrightarrow[\tau_1]{} & F' \end{array}$$

The first two commutative squares say that τ_1 is a monoid morphism, while the last two squares say that τ_1 is a comonoid morphism. By definition we conclude that τ_1 is a morphism of Frobenius algebras.

At last, if we start with a given morphism of Frobenius algebras, we must use the data provided by its action on the generating arrows and the above commutative squares to construct a monoidal natural transformation. With these steps we find again the same transformation τ, as wanted. $\natural$

We now include an additional requirement of symmetry to the monoidal category and extend the proof of Theorem 9.1 to our needs.

Theorem 9.2. Let χ be a free symmetric monoidal category over a commutative Frobenius algebra 1 with braiding γ. If $(\mathtt{M}, \otimes, e, \sigma)$ is any symmetric monoidal category, then there exists a natural isomorphism

$$\mathtt{SymMon}(\chi, \mathtt{M}) \simeq \mathtt{cFrob}_{\mathtt{M}}.$$

Proof. From previous considerations, for any symmetric monoidal functor $\tau\colon \chi \to \mathtt{M}$ we have $\tau\gamma = \sigma$ and we obtain a commutative Frobenius algebra $(\tau(1), \tau\mu, \tau\eta, \tau\delta, \tau\varepsilon)$.

From the proof of Theorem 9.1, any commutative Frobenius algebra in $\mathtt{M}$ can be used to define a unique monoidal functor $\chi \to \mathtt{M}$. Moreover, the braiding γ induces a symmetric structure to such fuctor. Therefore the chosen commutative Frobenius algebra uniquely defines a unique object in $\mathtt{SymMon}(\chi, \mathtt{M})$.

For arrows, given any monoidal natural transformation $\tau\colon S \Rightarrow S'$ between symmetric monoidal functors $S, S' \in \mathtt{SymMon}(\chi, \mathtt{M})$. Denote $F := S(1)$ and $F' := S'(1)$. By the last theorem we obtain a morphism of Frobenius algebras $\tau_1\colon F \to F'$ which trivially satisfies the commutativity of the square

$$\begin{array}{ccc} F \otimes F & \xrightarrow{\tau_2} & F' \otimes F' \\ {\scriptstyle S\gamma=\sigma}\downarrow & & \downarrow{\scriptstyle S'\gamma=\sigma} \\ F \otimes F & \xrightarrow[\tau_2]{} & F' \otimes F' \end{array} \tag{2}$$

This shows that τ_1 is in fact a morphism between commutative Frobenius algebras. Now if we start with a morphism in $\mathtt{cFrob}_{\mathtt{M}}$ we can—as described in the last proof—obtain a unique monoidal functor $\chi \to \mathtt{M}$. This monoidal functor will however be symmetric, due to the data provided by the additional commutative square Eq. (2). With this we prove the wanted natural equivalence of categories. ♮

Now our main result comes merely as a free corollary of the last theorem.

Corollary 9.3. There exists a natural isomorphism

$$2\text{-}\mathtt{TQFT}_k \simeq \mathtt{cFrob}_{\mathtt{Vect}_k}.$$

Proof. Notice that since $2\text{-}\mathtt{TQFT}_k = \mathtt{SymMon}(2\text{-}\mathtt{cob}, \mathtt{Vect}_k)$ we can apply Theorem 9.2 and obtain the wanted natural isomorphism. ♮

With this we reach the end of our hasty expedition through the deep forest of topological quantum field theory, of which we only ventured its borders. I'm grateful for the reader who spent the time to read until the very end. For those who got motivated to read further, a possible continuation would be [Lur09]—which I didn't use as a reference, but wish I could.

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